1. Consider the following vector fields \( \vec{F}(x, y, z) \). Are they conservative? If so, find a function \( f(x, y, z) \) so that \( \nabla f = \vec{F} \). If not, justify your response.

Conservative vector field test: a vector field \( \vec{F} \) is conservative if the curl is the zero vector.

\[
\nabla \times \vec{F} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_x & F_y & F_z
\end{vmatrix} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} - \left( \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \right) \hat{y} + \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z} = \vec{0}
\]

a) \( \vec{F}(x, y, z) = (yz, xz, xy + 2z) \)

\[
\begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
yz & xz & xy + 2z
\end{vmatrix} = (x - x, y - y, z - z) = \vec{0}
\]

The vector field is conservative, therefore, a potential function exists. To find it, we must find the necessary terms from each component (We neglect the constant for now, we’ll add it back later).

\[
\int F_x \, dx = \int yz \, dx = xyz
\]

\[
\int F_y \, dy = \int xz \, dy = xyz
\]

\[
\int F_z \, dz = \int xy + 2z \, dz = xyz + z^2
\]

We see that the necessary terms are \( xyz \) and \( z^2 \), therefore

The field is conservative and has potential function \( f(x, y, z) = xyz + z^2 + C \)
b) \( \vec{F}(x, y, z) = \langle y + e^x, x - \cos y, 4 + z \rangle \)

\[
\begin{vmatrix}
\dot{x} & \dot{y} & \dot{z} \\
\frac{\partial x}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial z}{\partial z} \\
y + e^x & x - \cos y & 4 + z
\end{vmatrix} = \langle 0 - 0, 0 - 0, 1 - 1 \rangle = \vec{0}
\]

The vector field is conservative, therefore, a potential function exists. To find it, we must integrate each component (We neglect the constant for now, we’ll add it back later).

\[
\begin{align*}
\int F_x \, dx &= \int (y + e^x) \, dx = xy + e^x \\
\int F_y \, dy &= \int (x - \cos y) \, dy = xy - \sin y \\
\int F_z \, dz &= \int (4 + z) \, dz = 4z + \frac{1}{2}z^2
\end{align*}
\]

We see that the necessary terms are \( xy, e^x, -\sin y, 4z, \) and \( \frac{1}{2}z^2 \), therefore:

The field is conservative and has potential function \( f(x, y, z) = xy + e^x - \sin y + 4z + \frac{1}{2}z^2 \)

c) \( \vec{F}(x, y) = \langle y, z^2, x \rangle \)

\[
\begin{vmatrix}
\dot{x} & \dot{y} & \dot{z} \\
\frac{\partial x}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial z}{\partial z} \\
y & z^2 & x
\end{vmatrix} = \langle -2z, 1, -1 \rangle
\]

This vector field is not conservative. Therefore, a potential function does not exist.
2. A toilet paper manufacturing company has increased their production. Unfortunately, this production increase has caused a major manufacturing error! As you move towards the center of any one toilet paper roll, the sheets get progressively more dense. The density of a toilet paper roll can be modeled using the following function:

\[ \rho(r) = \cos\left(\frac{\pi(r - 1)}{6}\right) + 1 \]

\( r \) is the radial distance away from the center of the roll (inside the center cardboard tube). The whole roll can be modeled as a cylinder with an outer radius of 6, and inner radius of 2 (the cardboard tube radius), and a height of 10.

(a) Without using a calculator, calculate the mass of the toilet paper roll if the density everywhere was just 1 (leave \( \pi \) in your answer)

The mass of the tube is equal to the density times the volume:

\[ m = \rho V = \rho \pi (r_{outer}^2 - r_{inner}^2) h = 320\pi \]

(b) Set up the triple integral to solve for the mass of a toilet paper roll. Neglect the weight of the inner cardboard tube for your calculation.

Cylindrical coordinates are most useful here since the figure is a cylinder. The mass is found by integrating the density with respect to \( dV \) (in cylindrical coordinates).

\[
\int_{0}^{10} \int_{0}^{2\pi} \int_{2}^{6} \left( \cos\left(\frac{\pi(r - 1)}{6}\right) + 1 \right) r dr d\theta dz
\]

(c) Without using a calculator, solve the integral (leave \( \pi \) in your answer)

\[
(10)(2\pi) \int_{2}^{6} \cos\left(\frac{\pi(r - 1)}{6}\right) r dr
\]

\[
20\pi \left[ \left(\frac{r^2}{6}\right) \bigg|_{2}^{6} + \int_{2}^{6} \cos\left(\frac{\pi(r - 1)}{6}\right) r dr \right]
\]
Integration by parts

\[ u = r, \quad v = \frac{6}{\pi} \sin \left( \frac{\pi}{6} (r - 1) \right) \]

\[
20\pi \left[ 18 - 2 + \left( \frac{6}{\pi} \sin \left( \frac{\pi}{6} (r - 1) \right) \right) \right]_2^6 - \int_2^6 \frac{6}{\pi} \sin \left( \frac{\pi}{6} (r - 1) \right) \, dr
\]

\[
20\pi \left[ 16 + \frac{36}{\pi} \sin \left( \frac{5\pi}{6} \right) - \frac{12}{\pi} \sin \left( \frac{\pi}{6} \right) \right] - \int_2^6 \frac{6}{\pi} \sin \left( \frac{\pi}{6} (r - 1) \right) \, dr
\]

\[
20\pi \left[ 16 + \frac{18}{\pi} - \frac{6}{\pi} + \frac{6^2}{\pi^2} \cos \left( \frac{\pi}{6} (r - 1) \right) \right]_2^6
\]

\[
20\pi \left[ 16 + \frac{12}{\pi} + \frac{36}{\pi^2} \cos \left( \frac{5\pi}{6} \right) \cos \left( \frac{\pi}{6} \right) \right]
\]

\[
20\pi \left[ 16 + \frac{12}{\pi} + \frac{36}{\pi^2} \left( - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \right) \right]
\]

\[
20\pi \left[ 16 + \frac{12}{\pi} + \frac{36\sqrt{3}}{\pi^2} \right]
\]

\[
320\pi + 240 - \frac{720\sqrt{3}}{\pi}
\]

3. The vector field \( \vec{F} = \langle 2xy + 2x + y^2, 2xy + 2y + x^2 \rangle \) is conservative. Find a potential function \( f \) for \( \vec{F} \) (a function with \( \nabla f = \vec{F} \))

\[
\vec{F} = \langle f_x, f_y \rangle = \langle 2xy + 2x + y^2, 2xy + 2y + x^2 \rangle
\]

\[
\int f_x = xy^2 + x^2 + xy^2 + C(y)
\]

\[
\int f_y = xy^2 + y^2 + yx^2 + C(x)
\]

Looking at all the terms and comparing with \( \vec{F} \), we know that \( C(y) = y^2 \) and \( C(x) = x^2 \), therefore the potential function is:

\[
f(x, y, z) = xy^2 + x^2 + xy^2 + y^2
\]
4. Find the work done by the force field below in moving an object from (1,1) to (2,4) (HINT: Check if the vector field is conservative).

\[ \vec{F}(x, y) = (6y^\frac{2}{3}, 9x\sqrt{y}) \]

First we need to check if this vector field is conservative

\[
\frac{\partial F_x}{\partial y} = 9\sqrt{y} \quad \text{and} \quad \frac{\partial F_y}{\partial x} = 9\sqrt{y}
\]

Since \( \frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \) we can say the vector field is conservative.

Now we can find a function \( f(x, y) \) such that \( \nabla f = \vec{F} \)

\[
\int F_x \, dx = \int 6y^\frac{2}{3} \, dx = 6xy^\frac{2}{3}
\]

\[
\int F_y \, dy = \int 9x\sqrt{y} \, dy = 6xy^\frac{3}{2}
\]

Thus our potential function is

\[ f(x, y) = 6xy^\frac{2}{3} + C \]

Now that we have a potential function we can use the Fundamental Theorem of Line Integrals to compute the work done in moving from (1,1) to (2,4)

\[
W = \int_C \vec{F} \cdot dr = f(x_2, y_2) - f(x_1, y_1) = f(2, 4) - f(1, 1) = (96 + C) - (6 + C)
\]

\[ W = 90 \text{ (units)} \]