1. Rewrite the function $y(t) = 4\cos(7t) - 4\sin(7t)$ in phase amplitude.

A) $4\sqrt{2}\cos(7t + \frac{\pi}{4})$
B) $4\sqrt{2}\cos(7t - \frac{\pi}{4})$
C) $4\cos(7t + \frac{\pi}{4})$
D) $4\sqrt{2}\cos(7t)$

The phase-amplitude form of function $A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta)$

$$R = \sqrt{A^2 + B^2}$$
$$\cos(\delta) = \frac{A}{R}$$
$$\sin(\delta) = \frac{B}{R}$$

In our case $\omega = 7, R = \sqrt{4^2 + (-4)^2} = 4\sqrt{2}$ and $\tan(\delta) = \frac{-4}{4}$ so $\delta = -\frac{\pi}{4}$

2. The differential equation $mu'' + 4u' + 8u = 0$ describes a mass-spring system. For what values of $m$ is the system underdamped?

A) $m < \frac{1}{2}$
B) $m < 2$
C) $m > \frac{1}{2}$
D) $m > 2$

The system is underdamped when its characteristic equation $mr^2 + 4r + 8 = 0$ has two complex roots.
This happens when the discriminant $42 - 4m \cdot 8 = 16 - 32m$ is negative.
So, $16 - 32m < 0$ thus $m > \frac{1}{2}$.
3. For what forcing frequency $\omega$ is it possible for the solution to $18y'' + 2y = 81 \cos(\omega t)$ to grow without bound?

A) $\omega = 3$

B) $\omega = 9$

C) $\omega = \frac{1}{3}$

D) $\omega = \frac{1}{9}$

This is possible when the system is in resonance, meaning the external frequency $\omega$ equals the resonance frequency

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2}{18}} = \frac{1}{3}$$

4. Consider the following statements

(i) If $Y_1$ is a particular solution to the DE

$$y'' + p(t)y' + q(t)y = g(t)$$

then $Y_2 = cY_1$ is also a solution to the differential equation, where $c$ is a constant

(ii) If $y_1$ and $y_2$ are solutions to

$$ay'' + by' + cy = g(t)$$

then $y_1 - y_2$ is a solution to the homogeneous differential equation $ay'' + by' + cy = 0$ where $a, b,$ and $c$ are constants.

(iii) The solution to

$$ay'' + by' + cy = 0$$

with initial conditions $y(0) = A$ and $y'(0) = B$, is unique on $t \in (-\infty, \infty)$ where $a, b,$ and $c$ are constants.

(iv) Two solutions to a linear second order ODE can cross each other.

Which of the above statements are always true?

A) (i), (iii) and (iv)

B) (iii) and (iv)

C) (ii) and (iii)

D) (ii), (iii) and (iv)
(i) This statement is False:
\[ Y_2'' + p(t)Y_2' + q(t)Y_2 = g(t) \]
\[ c(Y_1'' + p(t)Y_1' + q(t)Y_1) = cg(t) \]
\[ cg(t) \neq g(t) \text{ if } c \neq 0 \text{ not ALWAYS true} \]

(ii) This statement is True
\[ a(y_1 - y_2)'' + b(y_1 - y_2)' + c(y_1 - y_2) \]
\[ (ay_1'' + by_1' + cy_1) - (ay_2'' + by_2' + cy_2) = g(t) - g(t) = 0 \]

(iii) This statement is True:
The existence and uniqueness theorem states that constant functions are continuous.
The initial conditions also exist on the interval.

(iv) This statement is True:
For example, \( \sin(t) \) and \( \cos(t) \) solve the differential equation \( y'' + y = 0 \) and their plots cross each other.

5. Consider the following initial value problem \((t - 5)y'' + \csc(t)y' + y = e^t\), \(y'(4) = 1\) and \(y(4) = \pi\). What is the largest interval on which the initial value problem is guaranteed to exist?

   A) (0,5)
   B) (0, 2\(\pi\))
   C) \((\pi, 5)\)
   D) \((\pi,2\pi)\)

Rewrite the equation as
\[ y'' + \frac{\csc(t)}{(t - 5)}y' + \frac{1}{(t - 5)}y = \frac{e^t}{(t - 5)} \]

The coefficients exist everywhere except \(t = 5\) and \(t = 0, \pi, 2\pi\)
We need to find the interval that contains the point 4 and none of the points that don’t exist, which is (C)
6. Identify the form of the particular solution for $y'' - 16y = (t - 3)e^{-4t} + (4t + 3)\sin(2t)$

A) $(At + B)te^{-4t} + (Ct + D)t\sin(2t) + (Et + F)t\cos(2t)$
B) $(At + B)e^{-4t} + (Ct + D)\sin(2t) + (Et + F)\cos(2t)$
C) $(At + B)e^{-4t} + (Ct + D)t\sin(2t) + (Et + F)t\cos(2t)$
D) $(At + B)te^{-4t} + (Ct + D)\sin(2t) + (Et + F)\cos(2t)$

The particular solution $Y$ should be of the same form as the right side of the equation. We might need to multiply by a power of $t$ to avoid duplications with the complementary solution. That is, the terms of $Y$ should not be solutions to the homogeneous DE

$$y_c = C_1e^{4t} + C_2e^{-4t}$$

The term $(t - 3)e^{-4t}$ corresponds to the particular solution term (we multiply by $t$ to avoid these duplicate terms)

$(At + B)te^{-4t}$

While the term $(4t + 3)\sin(2t)$ corresponds to the particular solution term

$(Ct + D)\sin(2t) + (Et + F)\cos(2t)$

Combine these two solutions to get the form of the solution

7. Find the general solution for the differential equation $y'' + 4y' + 4y = 0$

First find the solution to the homogeneous equation

$$r^2 + 4r + 4 = 0$$

This has a repeated root of $r = -2$

Then, we multiply one of the solutions by $t$ to obtain

$$y = C_1e^{-2t} + C_2te^{-2t}$$
8. Which of the following plots represents a solution to the ODE $y'' + 7y' + 6y = 0$?

![Plots](image)

A) (ii) only  
B) (iii) only  
C) (i) and (iv) only  
D) (ii) and (iii) only

The characteristic equation

$$r^2 + 7r + 6 = 0$$

has roots $r = -1$ and $r = -6$  
The solution is

$$y = c_1 e^{-t} + c_2 e^{-6t}$$

The solution tends to 0 and it equals 0 at most once