The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

1. Let \( f(x) = 8x^2 + 5 \). Use the definition of a derivative as a limit to prove that \( f'(x) = 16x \).

   Show each step in your calculation and be sure to use proper terminology in each step of your proof.

   \[
   f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{[8(x + h)^2 + 5] - [8x^2 + 5]}{h}
   \]

   \[
   f'(x) = \lim_{h \to 0} \frac{8(x^2 + 2xh + h^2) + 5] - [8x^2 + 5]}{h} = \lim_{h \to 0} \frac{8x^2 + 16xh + 8h^2 + 5 - 8x^2 - 5}{h}
   \]

   \[
   f'(x) = \lim_{h \to 0} \frac{h(16x + 8h)}{h} = \lim_{h \to 0} (16x + 8h) = 16x
   \]

2. Compute the following limit:

   \[
   \lim_{x \to \infty} \frac{91 \sqrt{x} + 3}{5 - 7 \sqrt{x}}
   \]

   \[
   \lim_{x \to \infty} \frac{\sqrt{x}(91 + \frac{3}{\sqrt{x}})}{\sqrt{x}(\frac{5}{\sqrt{x}} - 7)}
   \]

   \[
   \frac{91 + \frac{3}{\sqrt{x}}}{\frac{5}{\sqrt{x}} - 7} = \frac{91 + 0}{0 - 7} = -13
   \]
3. Iodine-131 has a half life of eight days. What is the original mass in milligrams (mg) of a sample if it has decayed to a mass of 150 mg after 21 days?

\[ A = Ce^{kt} \]

\[ A(0) = Ce^{k \cdot 0} \]

\[ \frac{1}{2} C = Ce^{k \cdot 8} \]

\[ \frac{1}{2} = e^{k \cdot 8} \rightarrow k = \frac{\ln(\frac{1}{2})}{8} \]

\[ A = 150 = C e^{\left(\frac{\ln(\frac{1}{2})}{8}\right) t} \]

\[ C = \frac{150}{\exp\left(\frac{4}{8}\ln 0.5\right)} \text{ mg} \]

4. Write an equation for each horizontal asymptote on the graph of the following function. Use limits to justify your answer. We will learn l’Hospital’s Rule and other shortcuts for obtaining limits later. For now, you are not allowed to use these approaches.

\[ \frac{56e^{-5x} - 30}{7e^{-5x} + 10} \]

\[ \lim_{x \to \infty} \frac{56e^{-5x} - 30}{7e^{-5x} + 10} = \frac{56 \cdot 0 - 30}{7 \cdot 0 + 10} = -3 \]

A helpful hint for this problem is to think about the graph of \( y = e^{-5x} \).

5. Compute the following limits:

(a) \( \lim_{x \to 0} \frac{19x - 5\sin(x)}{2x} \)

\[ \lim_{x \to 0} \left( \frac{19x}{2x} - \frac{5\sin(x)}{2x} \right) = \lim_{x \to 0} \left( \frac{19}{2} - \frac{5 \sin(x)}{x} \right) = \frac{19}{2} - \frac{5}{2} = 7 \]
* Remember that \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \)
(b)\[
\lim_{x \to 0} \frac{e^{6x} - 1}{e^{3x} - 1}
\]
\[
\lim_{x \to 0} \frac{(e^{3x})^2 - 1}{e^{3x} - 1} = \lim_{x \to 0} \frac{(e^{3x} - 1)(e^{3x} + 1)}{e^{3x} - 1}
\]
\[
\lim_{x \to 0} e^{3x} + 1 = e^{3*0} + 1 = 1 + 1 = 2
\]

6. Determine whether the following statements are always true.

(a) A function which is continuous at point (a) must also be differentiable at (a).

This statement is false. A function which is differentiable at (a) must be continuous at (a), but not the other way around. For example, \( F(x) = |x - 2| \) is continuous but not differentiable at 2.

(b) If \( f(x) = \sin(x^3) \) and \( g(x) \) is an odd function, then the composite function \( g(f(x)) \) is an odd function.

This statement is true.

\[
L = w(x) = (g \ast f)(x) = w(-x) = g(f(-x)) = g(\sin((-x)^3)) = g(\sin(-x^3))
\]
\[
= g(-\sin(x^3)) = -g(\sin(x^3)) = -g(f(x)) = -w(x) \text{ because sine is odd and since } g \text{ is odd.}
\]

(c) If the finite limit \( \lim_{t \to 2} \frac{h(t) - h(2)}{t - 2} \)

This statement is true. \( H \) is differentiable at 2. \( H \) is also continuous at 2.

(d) The function \( y = \frac{9x-63}{x^2+6x-91} \) has a vertical asymptote at \( x = 7 \).

This statement is false.

\[
\lim_{x \to 7} \frac{9x-63}{x^2+6x-91} = \lim_{x \to 7} \frac{9(x-7)}{(x-7)(x+13)}
\]
\[
\lim_{x \to 7} \frac{9}{x+13} = \frac{9}{20}
\]

Clearly this does not go to +/- \( \infty \) so there is no asymptote.

(e) If the point \( (\frac{1}{4}, -4) \) is on the graph of a one-to-one function \( f(x) \), then the point \( (4, -\frac{1}{4}) \) must also be on the graph of \( f^{-1}(x) \).

This statement is false.

\( (\frac{1}{4}, -4) \) is on the graph of \( F(x) \) and \( (-4, \frac{1}{4}) \) is on the graph of \( F'(x) \) but not \( (4, -\frac{1}{4}) \)
7. Determine the x-intercept on the graph of the following function. Simplify your answer.

\[ f(x) = e^{9x} - 121e^{7x} \]

To find x-intercept, set \( f(x) = 0 \).

\[ f(x) = 0 \rightarrow e^{9x} - 121e^{7x} = 0 \]

\[ e^{9x} = 121e^{7x} \]

\[ \frac{e^{9x}}{e^{7x}} = 121 \]

\[ e^{2x} = 121 \]

Applying log on both sides of the equations...

\[ \ln(e^{2x}) = \ln(121) \]

\[ 2x = \ln(121) \rightarrow 2x = \ln(11^2) \]

\[ 2x = 2 \ln(11) \]

\[ x = \ln(11) \] which gives us what the x-intercept is

8. Evaluate the following limits and write your answers in simplified form.

(a)

\[ \lim_{x \to \sqrt{2}} \frac{120 \arcsin\left(\frac{x}{2}\right)}{x^2 + 4} \]

First, we can directly plug in the value of the limit:

\[ \frac{120\arcsin\left(\frac{\sqrt{2}}{2}\right)}{(\sqrt{2})^2 + 4} \]

\[ \frac{120\left(\frac{\pi}{4}\right)}{2 + 4} \]

\[ \frac{120\left(\frac{\pi}{4}\right)}{6} \]
$$20\left(\frac{\pi}{4}\right) = 5\pi$$

(b) 

$$\lim_{x \to \infty} 13 + 5 \sin(9e^{3x} + 6)x^{10}$$

Within this specific function, the range of sine fluctuates between -1 and 1:

$$-1 < \sin(9e^{3x} + 6) < 1$$: Now multiply by 5

$$-5 < 5 \sin(9e^{3x} + 6) < 5$$: Add 13 to all sides

$$8 < 13 + 5 \sin(9e^{3x} + 6) < 18$$: Divide all sides by $x^{10}$

$$\frac{8}{x^{10}} < \frac{13 + 5 \sin(9e^{3x} + 6)}{x^{10}} < \frac{18}{x^{10}}$$

$$\lim_{x \to \infty} \frac{8}{x^{10}} = 0$$ and $$\lim_{x \to \infty} \frac{18}{x^{10}} = 0$$

So, by the squeeze theorem, it must be true that:

$$\lim_{x \to \infty} \frac{13 + 5 \sin(9e^{3x} + 6)}{x^{10}} = 0$$