Optimization of the 2016 Spanish General Election

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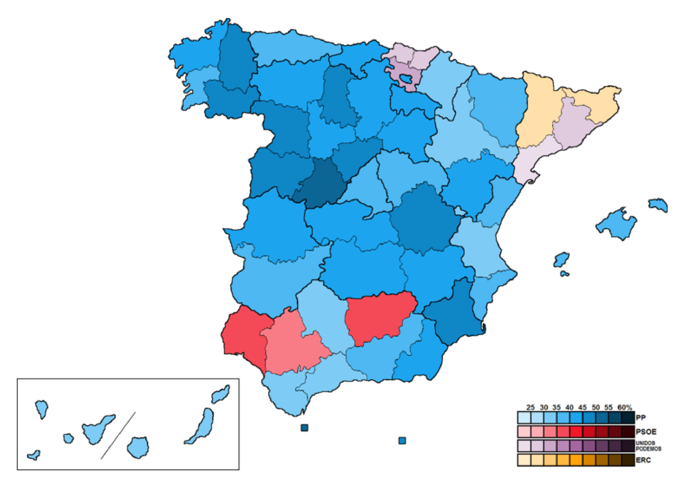
Introduction

The latest Spanish general election was held on Jun.26 2016 to elect the 12th *Cortes Generales del Reino de España*. All 350 seats in the Congress of Deputies were up for election within 19 constituencies (17 autonomic communities and 2 autonomic cities) in Spain. As a result, 9 of the parties obtained at least one seat out of 350.



The **Proportional Representation(PR)** rule was utilized in the Spanish General Election, where the **D'Hondt Law** $V_i/(S_i+1)$ was employed to allocate the seats corresponding to the number of votes. However, D'Hondt Law might have a poor performance in especially the small parties with fewer votes, leading to an inequity of seat acquisition; hence the overrepresentation or underrepresentation may occur.

Objectives

- **Optimization**: This research aims to reduce and minimize the disproportions (overrepresentation or underrepresentation) involved in the PR rules of the Spanish General Election.
- **Weighted Seats**: The number of reallocated and weighted seats corresponding to constituencies by party are expected to be calculated based on the established optimizations.

Election Results Breakdown

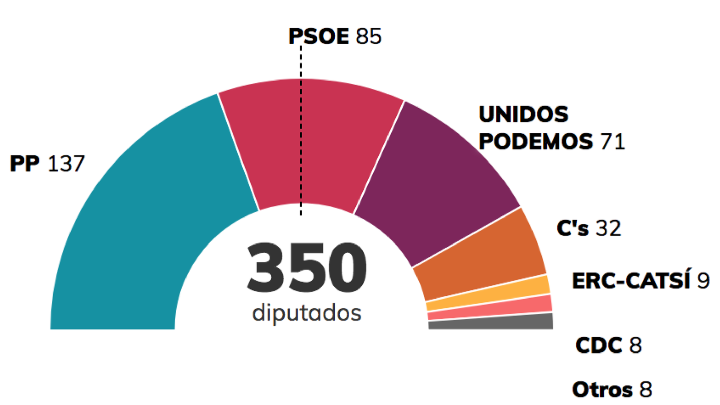


Figure 2: *The number of seats by party*

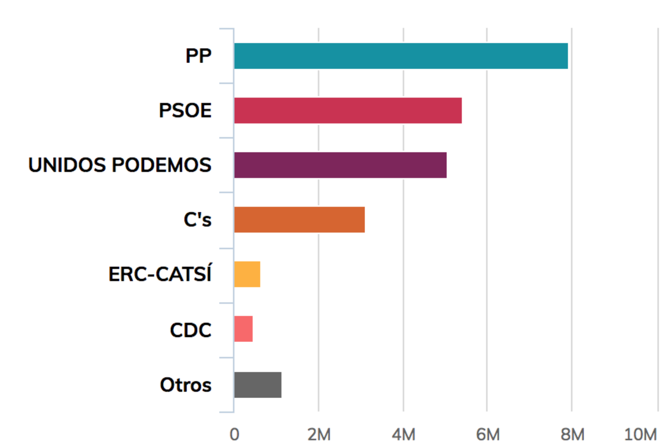


Figure 3: *The number of votes by party*

- **Figure 2** indicates the seats obtained by different parties, from which People' Party(PP) achieved the most number of seats(137), with Spanish Socialist Workers' Party (PSOE), United We Can (Podemos), Citizens' Party (C's), Catalonia Yes (ERC CatSí), and Democratic Convergence of Catalonia (CDC) following in order. There are in total 9 parties(3 minor parties included in the OTHER category) that have obtained seats.
- **Figure 3** shows the number of total votes by party, which roughly matches with the proportions in **Figure 2**, subject to the PR rules.

Methods

In order to optimize the electoral system, least square methods were employed to minimize the disproportions.

Notations Number of parties: P, Number of constituencies: R; Total number of seats: m; Total number of votes: v; Number of seats in constituency j: m_j; Number of votes of party i: v_i; Number of votes of party i in constituency j: $v_{(i,j)}$; Number of seats of party i in constituency j after the integer allocation of seats: $m_{(i,j)}$.

$$\begin{cases} \min\left(\left(x_{11}-\frac{m_1 v_{11}}{\sum_{i=1}^{P} v_{i1}}\right)^2+\cdots+\left(x_{P1}-\frac{m_1 v_{P1}}{\sum_{i=1}^{P} v_{i1}}\right)^2+\cdots+\left(x_{1R}-\frac{m_R v_{1R}}{\sum_{i=1}^{P} v_{iR}}\right)^2+\cdots+\left(x_{PR}-\frac{m_R v_{PR}}{\sum_{i=1}^{P} v_{iR}}\right)^2\right) \\ x_{11}+\cdots+x_{P1}=m_1 \\ \dots\dots\dots\dots \\ x_{1R}+\cdots+x_{PR}=m_R \\ x_{11}+\cdots+x_{1R}=\frac{m\, v_1}{v} \\ \dots\dots\dots\dots \\ x_{P1}+\cdots+x_{PR}=\frac{m\, v_P}{v} \\ x_{ij}\ge 0\ \forall i,j \end{cases}$$

To solve it, we have to find the critical points by taking the partial derivative of the corresponding functions. We also introduce the λs as the weighted coefficients of the optimization system.

$$\begin{aligned} &F(x_{11},\dots,x_{1R},\dots,x_{P1},\dots,x_{PR},\lambda_1,\dots,\lambda_R,\lambda_{R+1},\dots,\lambda_{R+P}) \\ &= \left(x_{11}-\frac{m_1 v_{11}}{\sum_{i=1}^{P} v_{i1}}\right)^2+\cdots+\left(x_{1R}-\frac{m_R v_{1R}}{\sum_{i=1}^{P} v_{iR}}\right)^2+\left(x_{P1}-\frac{m_1 v_{P1}}{\sum_{i=1}^{P} v_{i1}}\right)^2+\cdots \\ &+\left(x_{PR}-\frac{m_R v_{PR}}{\sum_{i=1}^{P} v_{iR}}\right)^2+\lambda_1\left(x_{11}+\cdots+x_{P1}-m_1\right)+\cdots+\lambda_R\left(x_{1R}+\cdots+x_{PR}-m_R\right) \\ &+\lambda_{R+1}\left(x_{11}+\cdots+x_{1R}-\frac{m\, v_1}{v}\right)+\cdots+\lambda_{R+P}\left(x_{P1}+\cdots+x_{PR}-\frac{m\, v_P}{v}\right) \end{aligned}$$

By taking the partial derivative of xs and λs, the local extremum of each single-value dimension will be found, which indicates the local minimum. Therefore, seeking for the local extremums is equivalent to solving the linear system shown below:

$$\left.\begin{aligned} F_{x_{11}} &= 2\left(x_{11}-\frac{m_1 v_{11}}{\sum_{i=1}^{P} v_{i1}}\right)+\lambda_1+\lambda_{R+1}=0 \\ &\dots\dots \\ F_{x_{1R}} &= 2\left(x_{1R}-\frac{m_R v_{1R}}{\sum_{i=1}^{P} v_{iR}}\right)+\lambda_R+\lambda_{R+1}=0 \\ &\dots\dots \\ F_{x_{P1}} &= 2\left(x_{P1}-\frac{m_1 v_{P1}}{\sum_{i=1}^{P} v_{i1}}\right)+\lambda_1+\lambda_{R+P}=0 \\ &\dots\dots \\ F_{x_{PR}} &= 2\left(x_{PR}-\frac{m_R v_{PR}}{\sum_{i=1}^{P} v_{iR}}\right)+\lambda_R+\lambda_{R+P}=0 \\ F_{x_{\lambda_1}} &= x_{11}+\cdots+x_{P1}-m_1=0 \\ &\dots\dots \\ F_{x_{\lambda_R}} &= x_{1R}+\cdots+x_{PR}-m_R=0 \\ &\dots\dots \\ F_{x_{\lambda_{R+1}}} &= x_{11}+\cdots+x_{1R}-\frac{m\, v_1}{v}=0 \\ &\dots\dots \\ F_{x_{\lambda_{R+P}}} &= x_{P1}+\cdots+x_{PR}-\frac{m\, v_P}{v}=0 \end{aligned}\right\}$$

$$\left.\begin{aligned} F_{x_{11}} &= 2\,x_{11}+\lambda_1+\lambda_{R+1}=\frac{m_1 v_{11}}{\sum_{i=1}^{P} v_{i1}} \\ &\dots\dots \\ F_{x_{1R}} &= 2\,x_{1R}+\lambda_R+\lambda_{R+1}=\frac{m_R v_{1R}}{\sum_{i=1}^{P} v_{iR}} \\ &\dots\dots \\ F_{x_{P1}} &= 2\,x_{P1}+\lambda_1+\lambda_{R+P}=\frac{m_1 v_{P1}}{\sum_{i=1}^{P} v_{i1}} \\ &\dots\dots \\ F_{x_{PR}} &= 2\,x_{PR}+\lambda_R+\lambda_{R+P}=\frac{m_R v_{PR}}{\sum_{i=1}^{P} v_{iR}} \\ F_{x_{\lambda_1}} &= x_{11}+\cdots+x_{P1}=m_1 \\ &\dots\dots \\ F_{x_{\lambda_R}} &= x_{1R}+\cdots+x_{PR}=m_R \\ &\dots\dots \\ F_{x_{\lambda_{R+1}}} &= x_{11}+\cdots+x_{1R}=\frac{m\, v_1}{v} \\ &\dots\dots \\ F_{x_{\lambda_{R+P}}} &= x_{P1}+\cdots+x_{PR}=\frac{m\, v_P}{v} \end{aligned}\right\}$$

In the linear system, total number of functions is ($P\times R+P+R$), with the unknown values of the same amount. That is to say, a squared coefficient matrix A will be generated to solve the linear system $AX=B$.

$$\begin{pmatrix} 20\dots00\dots00\dots010\dots010\dots0 & \frac{m_1 v_{11}}{\sum_{i=1}^{P} v_{i1}} \\ \dots\dots\dots\dots\dots\dots\dots\dots & \\ 0\dots020\dots00\dots00\dots0110\dots0 & \frac{m_R v_{1R}}{\sum_{i=1}^{P} v_{iR}} \\ 0\dots00\dots020\dots01\dots00\dots01 & \frac{m_1 v_{P1}}{\sum_{i=1}^{P} v_{i1}} \\ \dots & \\ 100\dots\dots.0001000..100\dots100\dots & m_1 \\ 010\dots\dots.0000100..010\dots010\dots & m_2 \\ \dots & \\ 0000\dots100\dots.100\dots.100\dots.1\dots & m_R \\ 111111\dots11000000\dots.0000\dots & \frac{m\, v_1}{v} \\ \dots & \\ 000000\dots00000000\dots.1111\dots & \frac{m\, v_P}{v} \end{pmatrix}$$

Matrix A

The coefficient matrix A of the linear system is shown left-side. As a result, it is always computationally singular since there are too many zeros existing in the matrix. To precisely return the final solutions of matrix B, it is more desirable to employ mathematical computing tools which can tolerate a higher precision, such as MATLAB and Mathematica.

Results

1. Practically, the values of all the $x_{(i,j)}$ should be positive. However, due to the singularity of the coefficient matrix, some of the $x_{(i,j)}$ values are negative (but very close to zero). In this way, we simply coerce the negative values to zero since they must originally have obtained zero seats by the D'Hondt Law.
2. For the unique solution of this system, we have that $(x_{(1,1)},\dots,x_{(1,R)},\dots,x_{(P,1)},\dots,x_{(P,R)})$ gives the weighted number of seats of party i in constituency j. The weight of each seat of party i in constituency j would be $x_{(i,j)}/m_{(i,j)}$ (given that each $m_{(i,j)}$ is not zero).
3. If the party i has been overrepresented(OR) in the integer allocation of seats for constituency j, then we have that $x_{(i,j)}/m_{(i,j)}<1$, if the party i has been underrepresented(UR) in the integer allocation of seats for constituency j, then we have that $x_{(i,j)}/m_{(i,j)}>1$ (given that each $m_{(i,j)}$ is not zero).

Discussion

- Taking the constituency Andalucía as an example, the original seats distributed to each party, $m_{(i,j)}$, are 23, 20, 11, 7, 0, 0, 0, 0, and 0. After being optimized, the weighted seats, $x_{(i,j)}$, are 16.01(OR), 14.3(OR), 10.256(OR), 7.8(UR), 2.634(UR), 2.57(UR), 2.4534(UR), 2.49(UR), 2.39(UR) respectively. We can easily judge that the variance has been reduced since intuitively the seats are distributed more evenly to the parties.
- This method can minimize the variance or disproportions involved in the seats allocation caused by the D'Hondt Law. However, such results would surely increase the bias of the electoral system.
- In this research, every party was treated without any discriminant. That is to say, every constituency has the same probability to vote for any party. Actually, it was not always such a thing. Some party is unique in a certain constituency and only this constituency would exclusively vote for it. In this way, the conditions are actually not evenly distributed for every constituency. The results will surely have some bias. Therefore, an evaluation standard to judge the trade-off is required to deal with specific issues.

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