



Center for Academic Resources in Engineering (CARE) Peer Exam Review Session

Math 241 — Calculus III

Midterm 1 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Feb 28, 5-6:30pm Nidhi and Danny Session 2: Mar 1, 7-8:30pm Pieter and David

Can't make it to a session? Here's our schedule by course:

<https://care.engineering.illinois.edu/tutoring-resources/tutoring-schedule-by-course/>

Solutions will be available on our website after the last review session that we host, as well as posted in the zoom chat 30 minutes prior to the end of the session

Step-by-step login for exam review session:

1. Log into [Queue @ Illinois](#)
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"
5. Join the zoom link in the staff message

Please do not log into the zoom call without adding yourself to the queue

Good luck with your exam!

1. Find $\vec{u} \times \vec{v}$ if $\vec{u} = \langle 3, -4, 1 \rangle$ and $\vec{v} = \langle 5, 2, -6 \rangle$

$$\begin{aligned}\vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -4 & 1 \\ 5 & 2 & -6 \end{vmatrix} = \begin{vmatrix} -4 & 1 \\ 2 & -6 \end{vmatrix} \hat{i} - \begin{vmatrix} 3 & 1 \\ 5 & -6 \end{vmatrix} \hat{j} + \begin{vmatrix} 3 & -4 \\ 5 & 2 \end{vmatrix} \hat{k} \\ &= 22\hat{i} - (-23)\hat{j} + 26\hat{k} \\ &= \boxed{\langle 22, 23, 26 \rangle}\end{aligned}$$

2. Find an equation for the plane that passes through the point $P = (1, 2, 3)$ and contains the line L given by the parametric equation:

$$x(t) = 1 - 3t$$

$$y(t) = 3$$

$$z(t) = 6 + 2t$$

for $-\infty < t < \infty$

We always need two pieces of information for the equation of a plane:

- Point on the plane, $P(x_0, y_0, z_0)$
- Vector normal to the plane, $\vec{N}$

The equation of a plane, then, is $0 = \vec{N} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle$

Let $P_0 = (1, 2, 3)$

From the given line equations (when $t = 0$):

$P_1(1, 3, 6)$

Direction vector of the line $\vec{a} = \langle -3, 0, 2 \rangle$

In order to compute a cross product and get the normal vector $\vec{N}$, we need one more vector. Subtract points P_1 and P_0 . This gives us another vector in the plane that we want between P_1 and P_0 .

We get the vector $\vec{b} = \langle 1 - 1, 3 - 2, 6 - 3 \rangle = \langle 0, 1, 3 \rangle$

Now, $\vec{N} = \vec{a} \times \vec{b} = \langle -2, 9, -3 \rangle$

To get the equation of the plane:

$$0 = \vec{N} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle$$

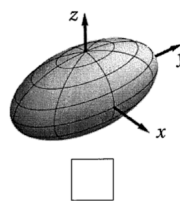
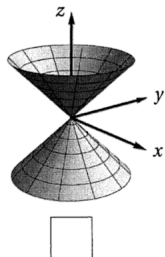
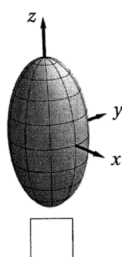
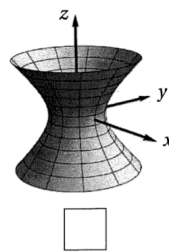
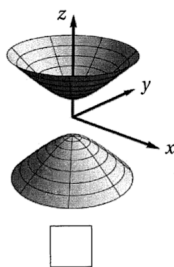
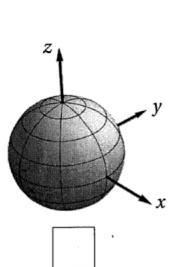
$$0 = \langle -2, 9, -3 \rangle \cdot \langle x - 1, y - 2, z - 3 \rangle$$

$$-2x + 9y - 3z = 7$$

3. For each equation below, write the corresponding letter in the box next to the picture of the surface it describes

(A) $x^2 + y^2 + z^2 + 1 = 0$

(B) $4x^2 + y^2 + 4z^2 - 1 = 0$



(A) is top middle (hyperboloid of two sheets)

(B) is bottom right (ellipsoid with longest axis along $\hat{y}$)

Here is a little cheat sheet for the equations of quadric surfaces:

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

4. Consider a function $f(t) = f(x(t), y(t), z(t)) = xyz - z^2$, where $x(t)$, $y(t)$, $z(t)$ are defined as followed:

$$x(t) = 2t^2 + 1$$

$$y(t) = 3 - \frac{1}{t}$$

$$z(t) = 3$$

Find the following values:

(a) $f_z(3, 1, 2)$

$$\frac{\partial}{\partial z}(xyz - z^2)$$

$$xy - 2z$$

$$f_z(3, 1, 2) = 3 - 4 = -1$$

(b) $\left. \frac{dx}{dt} \right|_{(t=0)}$

$$\frac{d}{dt}x = 4t$$

$$\left. \frac{dx}{dt} \right|_{(t=0)} = 0$$

(c) $\left. \frac{df}{dt} \right|_{(t=1)}$

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} = yz(4t) + xz\left(\frac{1}{t^2}\right) + (xz - 2z)(0)$$

When $t = 1$:

$$x = 3$$

$$y = 2$$

$$z = 3$$

Plug into the equation:

$$\left. \frac{df}{dt} \right|_{(t=1)} = 24 + 9 + 0 = 33$$

5. Consider two points A and B .

$$\begin{aligned} A &= (0, 7, 2) \\ B &= (1, 2, 0) \end{aligned}$$

(a) Find the vector that represents the displacement between the points (vector drawn from A to B)

$$\vec{s} = B - A = \boxed{\langle 1, -5, -2 \rangle}$$

(b) What is the projection of this vector onto $\vec{r} = \langle 5, -2, 7 \rangle$?

To project onto $\vec{r} = \langle 5, -2, 7 \rangle$ we use the projection formula

$$\text{proj}_{\vec{r}} \vec{s} = \left(\frac{\vec{r} \cdot \vec{s}}{\vec{r} \cdot \vec{r}} \right) \vec{r}$$

$$\boxed{\left\langle \frac{5}{78}, \frac{-2}{78}, \frac{7}{78} \right\rangle}$$

(c) What is the projection of the vector from part (a) onto the plane represented by the equation $5x - 2y + 7z = 10$

Notice that the normal vector of the plane is the same as vector $\vec{r}$ in part (b). To project $\vec{s}$ onto the plane, we can do an orthogonal projection of $\vec{s}$ onto the plane's normal vector $\vec{r}$ and then do vector subtraction to find the orthogonal component (which is what we want).

$$\text{orth}_{\vec{r}} \vec{s} = \vec{s} - \text{proj}_{\vec{r}}(\vec{s}) = \langle 1, -5, -2 \rangle - \left\langle \frac{5}{78}, \frac{-2}{78}, \frac{7}{78} \right\rangle$$

$$\boxed{\left\langle \frac{73}{78}, -\frac{194}{39}, -\frac{163}{78} \right\rangle}$$

6. Consider the limit

$$\lim_{(x,y) \rightarrow (0,0)} \frac{y^4 \cos^2 x}{x^4 + y^4}$$

Does this limit exist? If so, what is its value? Justify your answer.

Along $x = 0$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{y^4 \cos^2 x}{x^4 + y^4} = \lim_{(x,y) \rightarrow (0,0)} \frac{y^4}{y^4} = 1$$

Along $y = 0$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{y^4 \cos^2 x}{x^4 + y^4} = \lim_{(x,y) \rightarrow (0,0)} \frac{0^4 \cos^2 x}{x^4 + 0^4} = 0$$

Since $0 \neq 1$, the Limit DNE.

7. Determine if the two lines intersect. If so, find their point of intersection.

$$\begin{cases} x = 3t - 3 \\ y = -2t + 1 \\ z = 4t - 2 \end{cases} \quad \begin{cases} x = 2s + 3 \\ y = 2s - 1 \\ z = s + 2 \end{cases}$$

If these two lines intersect, there will be a value $t = t_0$ and $s = s_0$ (not necessarily the same), such that each component of the first line equals each component of the second line. Thus we can equate the x and y components of each one, solve for t and s , and see if these values also satisfy the z component.

$$\begin{aligned} 3t - 3 &= 2s + 3 \\ -2t + 1 &= 2s - 1 \end{aligned}$$

Solving these two gives $t = \frac{8}{5}$ and $s = -\frac{3}{5}$. We can plug these into the equations for z to see if they will also match up.

$$\begin{aligned} z &= 4\left(\frac{8}{5}\right) - 2 = \frac{22}{5} \\ z &= -\frac{3}{5} + 2 = \frac{7}{5} \end{aligned}$$

These lines do not intersect

8. Find the equation of the line of intersection of the following two planes: $2x + y - 2z = 2$ and $-2x + y + z = 6$.

We are looking for the equation of a line, which means we need a starting point and a direction vector. Finding a starting point is easy, we just need a point that is on both planes. To do this, we will let $z = 0$ and solve the two equations for x and y .

$$\begin{aligned} 2x + y &= 2 \\ -2x + y &= 6 \end{aligned}$$

Solving gives a point on both planes $(-1, 4, 0)$. This will be our starting vector.

Now we need a direction vector. Notice that this vector will be perpendicular to each plane's normal vector. Therefore, we can take a cross product.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ -2 & 1 & 1 \end{vmatrix} = \langle 3, 2, 4 \rangle$$

Therefore, the equation of the line of intersection is ¹

$$\boxed{\vec{r}(t) = \langle -1 + 3t, 4 + 2t, 4t \rangle}$$

9. Explain why $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$.

$\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. The dot product of perpendicular vectors is zero.

10. Describe in words what the contour lines of a function $f(x, y)$ would look like at a point (a, b) if, at that point, $\frac{\partial f}{\partial x} > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$

The contour lines would be increasing in their z value, but would be more spread out as you moved along the x axis.

¹Other answers are possible depending on which starting vector you use, but the direction vector should be parallel to this one